State Estimation

1.1 Base Orientation

1.1.1 Modeling

System:

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\omega}}^m \tag{1}$$

Measurement:

$$\boldsymbol{R}\boldsymbol{a}^m = \boldsymbol{g} \tag{2}$$

1.1.2 Complementary Filter

Predict:

$$\boldsymbol{R}_{k+1}^- = \boldsymbol{R}_k^+ \mathrm{Exp}\big(\boldsymbol{\omega}_k^m \Delta t\big) \tag{3}$$

Update:

$$\boldsymbol{R}_k^+ = \mathrm{Exp}\left(\alpha\delta\phi_k \boldsymbol{n}_k\right) \boldsymbol{R}_k^- \tag{4}$$

$$\boldsymbol{g}_k^- = \boldsymbol{R}_k^- \boldsymbol{a}_k^m \tag{5}$$

$$\delta\phi_k = \arccos\left(\frac{\boldsymbol{g}_k^-}{\|\boldsymbol{g}_k^-\|}\cdot\boldsymbol{z}\right) \tag{6}$$

$$\boldsymbol{n}_k = \begin{cases} \frac{1}{\sin\delta\phi_k}\frac{\boldsymbol{g}_k^-}{\|\boldsymbol{g}_k^-\|}\times\boldsymbol{z} & \text{if } \sin\delta\phi_k \neq 0 \\ \text{arbitrary} & \text{if } \sin\delta\phi_k = 0 \end{cases} \tag{7}$$

Notes:

- Update only when IMU is static (e.g., $\|\boldsymbol{a}^m\|$ is close to $\|\boldsymbol{g}\|$ for some period), i.e., the accelerometer measures gravity exclusively.
- To enhance smoothness, the parameter α can depend on the difference between $\|\boldsymbol{a}^m\|$ and $\|\boldsymbol{g}\|$.
- The unit vector $\boldsymbol{z} = [0, 0, \pm 1]^\top$ points along the gravity and its sign depends on the accelerometer reading.

1.2 Base Position & Velocity

1.2.1 Modeling

System:

$$\dot{\boldsymbol{p}} = \boldsymbol{v} \tag{8}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{a} = \boldsymbol{R}\left(\boldsymbol{a}^m - \boldsymbol{b} - \boldsymbol{n}^a\right) - \boldsymbol{g}, \quad \boldsymbol{n}^a \sim \mathcal{N}\big(\boldsymbol{0}, \boldsymbol{Q}^a\big) \tag{9}$$

$$\dot{\boldsymbol{b}} = \boldsymbol{n}^b, \quad \boldsymbol{n}^b \sim \mathcal{N}\big(\boldsymbol{0}, \boldsymbol{Q}^b\big) \tag{10}$$

$$\dot{\boldsymbol{c}}^i = \boldsymbol{n}^c, \quad \boldsymbol{n}^c \sim \mathcal{N}\big(\boldsymbol{0}, \boldsymbol{Q}^c\big) \tag{11}$$

Measurement:

$$\boldsymbol{z}^{p,i} = \boldsymbol{R}^\top\big(\boldsymbol{c}^i - \boldsymbol{p}\big) + \boldsymbol{n}^p, \quad \boldsymbol{n}^p \sim \mathcal{N}\big(\boldsymbol{0}, \boldsymbol{Q}^p\big) \tag{12}$$

$$\boldsymbol{z}^{v,i} = -\widehat{\boldsymbol{\omega}}\boldsymbol{R}^\top\big(\boldsymbol{c}^i - \boldsymbol{p}\big) - \boldsymbol{R}^\top\boldsymbol{v} + \boldsymbol{n}^v, \quad \boldsymbol{n}^v \sim \mathcal{N}\big(\boldsymbol{0}, \boldsymbol{Q}^v\big) \tag{13}$$

1.2.2 Kalman Filter

Process Model:

$$\underbrace{\left[\begin{array}{c} \boldsymbol{p}_{k+1} \\ \boldsymbol{v}_{k+1} \\ \hline \boldsymbol{b}_{k+1} \\ \hline \boldsymbol{c}^1_{k+1} \\ \vdots \\ \boldsymbol{c}^n_{k+1} \end{array}\right]}_{\boldsymbol{x}_{k+1}} = \underbrace{\left[\begin{array}{cc|c|ccc} \mathbb{I} & \Delta t\mathbb{I} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \mathbb{I} & -\Delta t\boldsymbol{R}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \mathbb{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathbb{I} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \mathbb{I} \end{array}\right]}_{\boldsymbol{\Phi}_k} \underbrace{\left[\begin{array}{c} \boldsymbol{p}_k \\ \boldsymbol{v}_k \\ \hline \boldsymbol{b}_k \\ \hline \boldsymbol{c}^1_k \\ \vdots \\ \boldsymbol{c}^n_k \end{array}\right]}_{\boldsymbol{x}_k} + \underbrace{\left[\begin{array}{c} \boldsymbol{0} \\ \Delta t\mathbb{I} \\ \hline \boldsymbol{0} \\ \hline \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{array}\right]}_{\boldsymbol{B}_k} \underbrace{(\boldsymbol{R}_k\boldsymbol{a}^m_k - \boldsymbol{g})}_{\boldsymbol{u}_k} + \tag{14}$$

$$\underbrace{\left[\begin{array}{c|c|ccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ -\boldsymbol{R}_k & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hline \boldsymbol{0} & \mathbb{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \mathbb{I} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \mathbb{I} \end{array}\right]}_{\boldsymbol{\Gamma}_k} \underbrace{\left[\begin{array}{c} \boldsymbol{n}^a_k \\ \hline \boldsymbol{n}^b_k \\ \hline \boldsymbol{n}^c_k \\ \vdots \\ \boldsymbol{n}^c_k \end{array}\right]}_{\boldsymbol{w}_k} \tag{15}$$

Measurement Model:

$$
\underbrace{\begin{bmatrix} \boldsymbol{z}_k^{p,1} \\ \vdots \\ \boldsymbol{z}_k^{p,n} \\ \hline \boldsymbol{z}_k^{v,1} \\ \vdots \\ \boldsymbol{z}_k^{v,n} \end{bmatrix}}_{\boldsymbol{z}_k} = \underbrace{\left[\begin{array}{cc|ccc} -\boldsymbol{R}_k^\top & \boldsymbol{0} & \boldsymbol{R}_k^\top & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{R}_k^\top & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{R}_k^\top \\ \hline \widehat{\boldsymbol{\omega}}_k \boldsymbol{R}_k^\top & -\boldsymbol{R}_k^\top & -\widehat{\boldsymbol{\omega}}_k \boldsymbol{R}_k^\top & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{\boldsymbol{\omega}}_k \boldsymbol{R}_k^\top & -\boldsymbol{R}_k^\top & \boldsymbol{0} & \cdots & -\widehat{\boldsymbol{\omega}}_k \boldsymbol{R}_k^\top \end{array}\right]}_{\boldsymbol{H}_k} \underbrace{\begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{v}_k \\ \hline \boldsymbol{c}_k^1 \\ \vdots \\ \boldsymbol{c}_k^n \end{bmatrix}}_{\boldsymbol{x}_k} + \underbrace{\begin{bmatrix} \boldsymbol{n}_k^p \\ \vdots \\ \boldsymbol{n}_k^p \\ \hline \boldsymbol{n}_k^v \\ \vdots \\ \boldsymbol{n}_k^v \end{bmatrix}}_{\boldsymbol{\nu}_k} \tag{16}
$$

Prediction:

$$\boldsymbol{x}_{k+1}^- = \boldsymbol{\Phi}_k \boldsymbol{x}_k^+ + \boldsymbol{B}_k \boldsymbol{u}_k \tag{17}$$

$$\boldsymbol{P}_{k+1}^- = \boldsymbol{\Phi}_k \boldsymbol{P}_k^+ \boldsymbol{\Phi}_k^\top + \boldsymbol{\Gamma}_k \boldsymbol{Q}_k \boldsymbol{\Gamma}_k^\top \tag{18}$$

Update:

$$\boldsymbol{x}_k^+ = \boldsymbol{x}_k^- + \boldsymbol{K}_k \big(\boldsymbol{z}_k - \boldsymbol{H}_k \boldsymbol{x}_k^- \big) \tag{19}$$

$$\boldsymbol{K}_k = \boldsymbol{P}_k^- \boldsymbol{H}_k^\top \big(\boldsymbol{H}_k \boldsymbol{P}_k^- \boldsymbol{H}_k^\top + \boldsymbol{V}_k \big)^{-1} \tag{20}$$

$$\boldsymbol{P}_k^+ = \big(\mathbb{I} - \boldsymbol{K}_k \boldsymbol{H}_k \big) \boldsymbol{P}_k^- \big(\mathbb{I} - \boldsymbol{K}_k \boldsymbol{H}_k \big)^\top + \boldsymbol{K}_k \boldsymbol{V}_k \boldsymbol{K}_k^\top \tag{21}$$

Notes:

- Update only when there is at least one foot contact.
- Update only with the measurements from the foot in contact, e.g., the measurement model may contain information of only two contacts.
- When new foot contact occurs, reset the previous a posteriori state estimate and covariance matrix (e.g., the foot contact part) for better performance.